Non-Linear Constitutive Equations for Sand under Cyclic Loading and Application to Geotechnical Earthquake Engineering

S. A. Savidis, D. Aubram, F. Rackwitz, W. Schepers

Soil Mechanics and Foundation Engineering Division
Technical University of Berlin, Germany

Abstract

Two advanced constitutive relations are introduced to model the strong non-linear material behavior of the sandy soil continuum. The first sand model, the *CSSA*-Model, is based on the critical state concept as well as on the bounding surface plasticity framework and the concept of generalized plasticity. It uses a state dependent dilatancy approach as a special feature. The second constitutive model is assigned to the hypoplastic framework. The material models were implemented into the implicit FE-code *ANSYS* which has been used for the analyses of element tests. Both, the *CSSA*-Model and the hypoplastic model, are capable to describe the particular sand behavior over a wide range of density and stress states with only one set of material parameters for each model. Especially the critical state sand model predicts the material response under cyclic loading conditions very satisfactorily. Hence, the *CSSA*-Model is particularly suitable for the application to the one-dimensional shear wave propagation.

INTRODUCTION

The realistic simulation of shear wave propagation in sands requires an appropriate constitutive equation for the granular material. Because of the fact that the mechanical behavior of sand depends on the density state, the stress state and the stress history as well as on the drainage situation, the continuum mechanical approach is rather complicated. Due to shear loading under the same stress state, the volume of a dense sand sample dilates, whereas a loose sand contracts. Irrespective of the current void ratio, sand exhibits an isochoric response if shearing continues, which is well-known as critical state. Additionally, sand is exposed to liquefaction under cyclic loading with undrained conditions.

Classical soil models treat the diverse appearances of sand behavior as different materials, although it is still the same sand. In the following two different models are presented. Both simulate the mechanical behavior of sand with a single set of parameters quite well, by taking the density as a state variable into account.

THE CSSA-MODEL

Model Framework

To account for the above mentioned facts of sand behavior, the critical state sand model (*CSSA*-Model) developed by Li (2000) incorporates a state dependent dilatancy. There are three concepts in the history of material modelling which affected the development of the elastic-plastic *CSSA*-Model. The well known Critical State framework (Roscoe *et al.* 1958) is effectively the basic idea. To model the response of sand under cyclic loading, the Bounding Surface Plasticity concept developed by Dafalias (1986) has been implemented. The Generalized Plasticity framework extended by Pastor *et al.* (1990) circumvents the difficulty to formulate the hardening and softening functions precisely by describing only the rate of these functions, also referred to as the plastic modulus.

Consider three specimen of the same sand with the same loading history and starting from the same stress state, but with different densities. If a shear loading increment from the same stress ratio $\eta = q/p'$ is applied, the loose

sand contracts and the dense sand dilates, like shown in the triaxial $q-p'$ plane in Fig 1, where $q=\sigma_1'-\sigma_3'$, $p'=(\sigma_1'+2\sigma_3')/3$ and σ_1' and σ_3' are the axial and the radial effective stress, respectively. In addition, medium-dense and dense sand passes a so-called phase transformation state (Ishihara *et al.* 1975, Li and Dafalias 2000). At this point the dilatancy D is equal to zero, although the sand is not at a critical state.
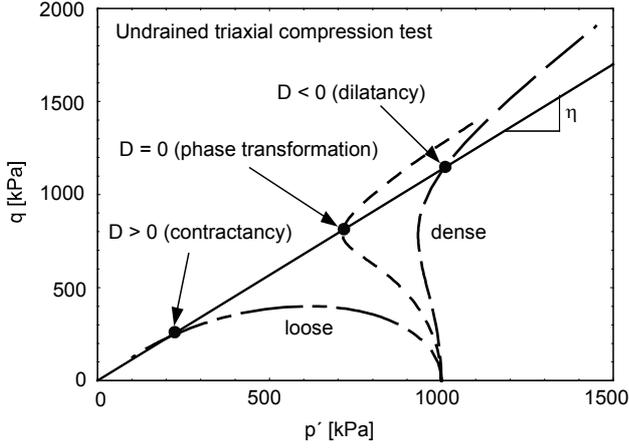

Fig 1: Undrained response of a sand with different densities under the same initial confining pressure

Li (1998) has shown that there exists a linear representation of the critical state line (*CSL*) in the $e-(p'/p_a)^\xi$ plane, where e is the void ratio, p_a is the atmospheric pressure and ξ is a material parameter (Fig 2). The internal state parameter ψ can be defined from Fig 2 as

$$\psi = e - e_c = e - \left[e_\Gamma - \lambda_c \left(p'/p_a \right)^\xi \right], \qquad (1)$$

where e_Γ and λ_c are material parameters.

One of the main features of the *CSSA*-Model is the state dependent dilatancy, due to the fact that the dilatancy for sand depends on the stress ratio η as well as on the void ratio e. The dilatancy function D_1 for the first bounding surface (Fig 3) has the form

$$D_1 = \frac{d_1}{M_c g(\theta)} \left[M_c g(\theta) e^{m\psi} \sqrt{\frac{\overline{\rho_1}}{\rho_1}} - R \right], \qquad (2)$$

where d_1, M_c, m are material parameters. The *CSSA*-Model includes 19 parameters - 15 material parameters determined from drained and undrained triaxial tests plus 4 default model constants. $\rho_1, \overline{\rho_1}$ are the mapping distances referred to the Bounding Surface Plasticity framework. The function $g(\theta)$, where θ is the Lode angle, establishes the shape of the first bounding surface and R is an invariant of the stress ratio tensor $r_{ij} = s_{ij}/p'$, where s_{ij} is the deviatoric stress tensor.
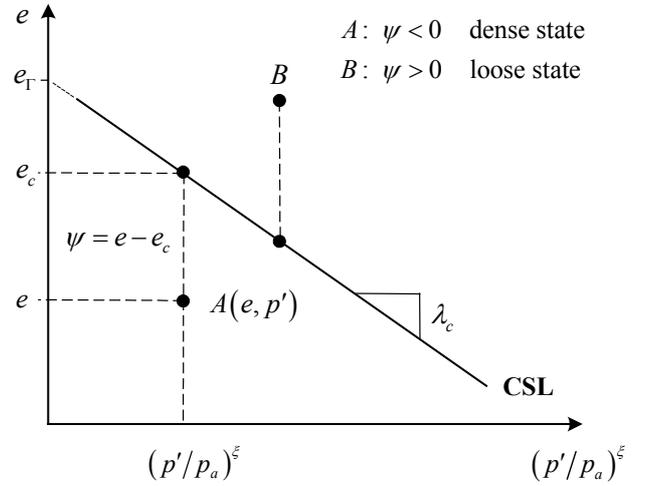

Fig 2: Critical State Line (*CSL*) and state parameter

The *CSSA*-Model defines two bounding surfaces, F_1 and F_2 (Fig 3). Bounding surface F_1 forms a cone around the hydrostatic axis p' restricted by a flat cap which represents the second bounding surface F_2. The current stress state is always located on a loading surface.
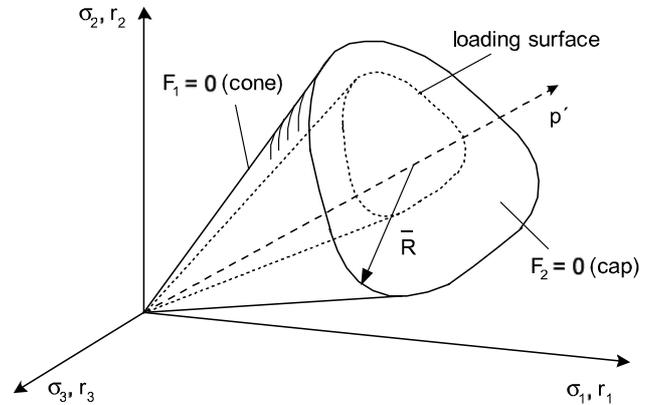

Fig 3: Bounding surfaces and loading surfaces of the *CSSA*-Model

Numerical Implementation

Because most of the geotechnical problems can only be solved numerically, a numerical representation of the soil model is needed. In a finite element code, the iteration of the global equilibrium is often done by a Newton-Raphson scheme (Fig 4). The local stress update, also referred to as the stress point algorithm, can be derived by an Euler method.
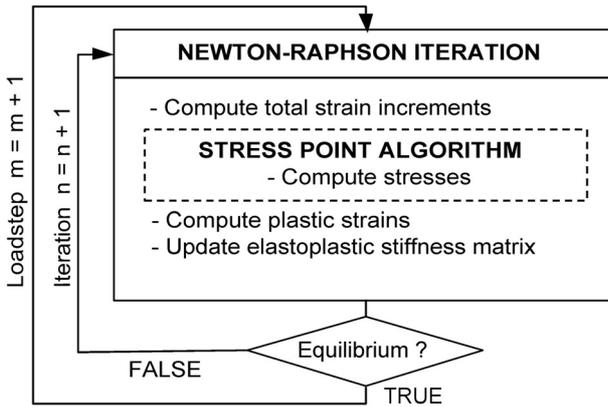

Fig 4: Newton-Raphson iteration of the global equilibrium

There are two groups of stress point algorithms. The explicit stress point algorithms are formulations using well known terms at the beginning of an increment, like the forward Euler scheme. Implicit stress point algorithms are formulations using terms at the end of an increment, like the backward Euler scheme. As a matter of fact, implicit algorithms usually require a specific formulation of the yield surface. Therefore, the application to generalized plasticity models, especially to the *CSSA*-Model, needs further investigations. The *CSSA*-Model has been implemented into the general purpose FE-code *ANSYS* using the material models interface *usermat* (Rackwitz (2003), ANSYS Inc. (2000)).

Numerical Simulation of Element Tests

The implementation of the *CSSA*-Model in *ANSYS* has been tested by simulation of element tests (Rackwitz (2003), Aubram (2004)). For the following numerical simulation of element tests the material parameters for Toyoura sand according to Li (2002) has been used. Fig 5 shows the simulation of an undrained triaxial compression test with Toyoura sand and four different initial confining

pressures in the $q - p'$ stress plane (Fig 5 top) and stress – axial strain plane (Fig 5 bottom).

The solid line show the results of the original stress point algorithm according to Li (1999). In this integration scheme, the size of the strain increments given by the global equilibrium iteration remains unchanged and the stress increment results from an iteration. The circle marks denote an alternative explicit algorithm (Aubram (2004)), in which the coarse strain increments are devided into a number of substeps defined by the user. However, the results of both algorithms coincide in this simulation.
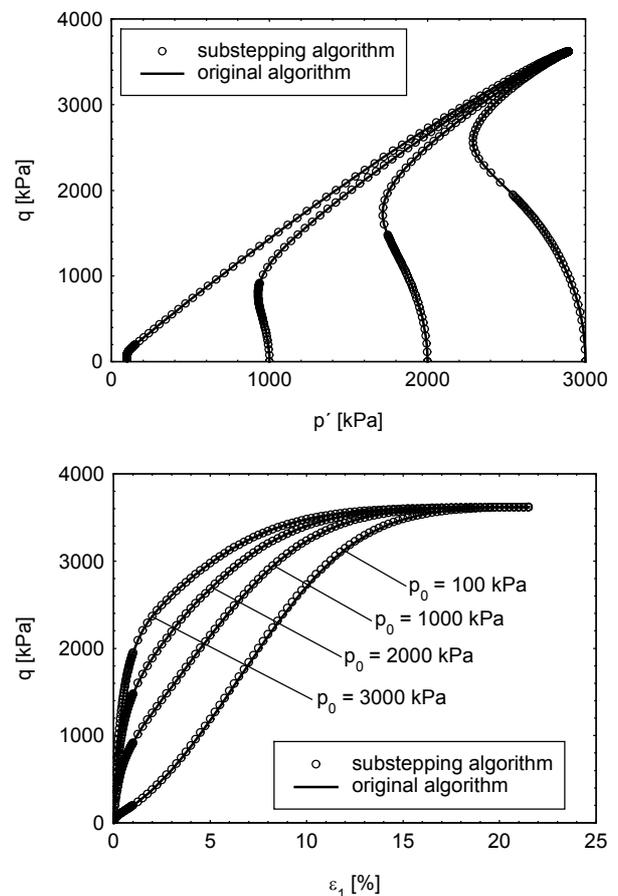

Fig 5: Simulation of undrained triaxial compression tests, Toyoura sand, D_r = 65%

Simulations of the experimental results of monotonic triaxial tests using the *CSSA*-Model are quite well, which has been presented by Li and Dafalias (2000). In contrast to that, the results of the simulation of a cyclic undrained triaxial test differ slightly, dependent on the size of the integration increments, as shown in the

$q - p'$ stress plane and stress – shear strain plane in Fig 6. Since the explicit substepping algorithm is only exact if the increments are infinite small, the calculation error accumulates along with the number of load steps.

Notwithstanding this fact, the *CSSA*-Model again provides good results. If the number of loading cycles increases, starting from isotropic consolidation, the effective mean normal stress decreases at constant density as a result of a rearrangement of the particles. In the region where p' vanishes, the sand begins to liquefy, staying within the bounding surfaces.
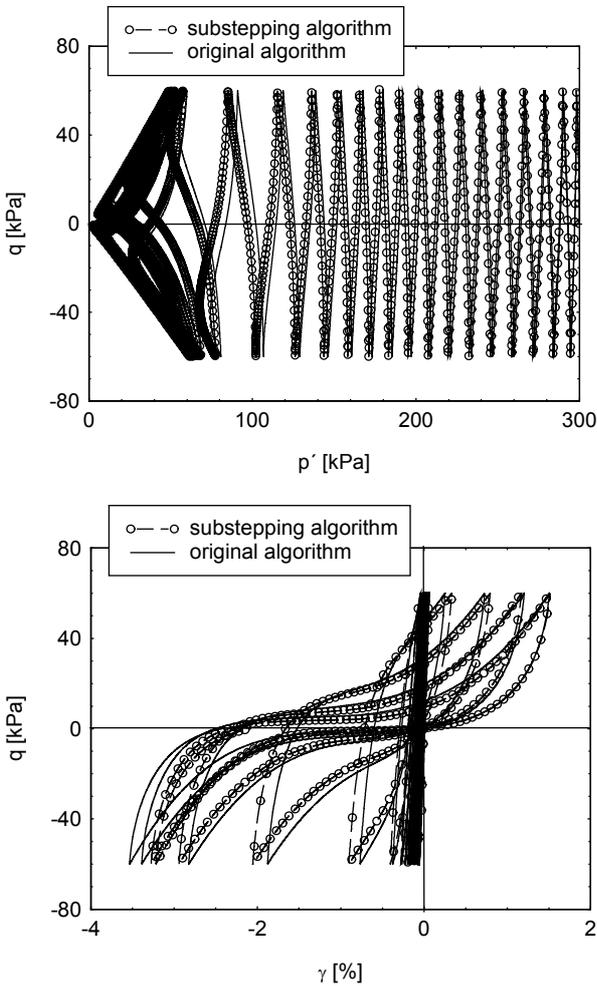

Fig 6: Simulation of an cyclic undrained triaxial test, Toyoura sand, D_r = 65%

Focussing on bounding surface F_2 on the hydrostatic axis, the *CSSA*-Model also formulates the response of sand under oedometric conditions, as shown in Fig 7, where ε_1 is the axial strain. Although the

current model version captures the cyclic behaviour under these conditions only rudimentarily, the *CSSA*-Model simulates the hysteresises under repeated loading and unloading quite well.
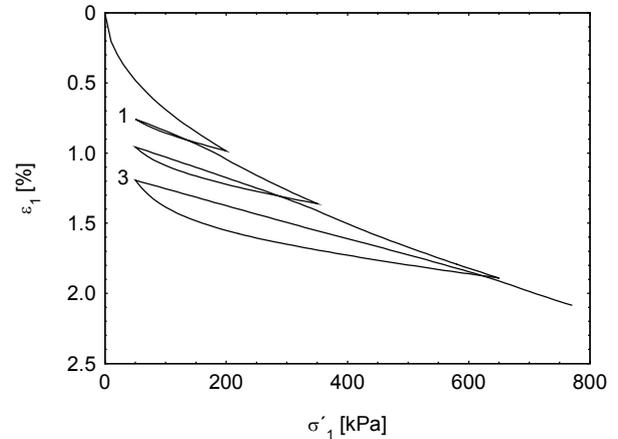

Fig 7: Simulation of an cyclic oedometer test, Toyoura sand, D_r = 65%

The results show that the *CSSA*-Model simulates the behavior of sand under monotonic and cyclic loading over a wide range of densities and stress states using a single set of parameters. Nevertheless, the results depend on the implemented stress point algorithm and react sensitively to the size of the calculation increments. It seems reasonable to implement other stress point algorithms, especially implicit algorithms in further investigations and to check and modify the formulation of the plastic moduli of the two bounding surfaces to enhance the model response to boundary value problems.

THE HYPOPLASTIC MODEL

Model Framework

The framework of hypoplasticity according to Kolymbas (2000), and Kolymbas *et al.* (1995, 1993) is an alternative and completely different approach compared to "classical" elastoplasticity. Hypoplasticity does not distinguish between elastic and plastic strains and it neither uses any yield or plastic potential surface nor normality and consistency rules. The name "hypoplastic" was taken from the relation between hypoplasticity and classical (elasto)plasticity on one side and hypoelasticity and elasticity on the other side. Both theories

with the prefix "hypo" do not require the use of a potential (Kolymbas *et al.* (1995, 1993)). The general constitutive equation for that class of hypoplastic models is of the form

$$\dot{T} = h\left(T, D, e\right),\qquad(3)$$

where $\dot{T}$ is the objective stress rate tensor as a functional h of the Cauchy stress tensor T, the deformation rate tensor D and the void ratio e.

The particular hypoplastic model according to von Wolffersdorff (1996), which has been used here, has the form

$$\dot{T} = f_b f_e \frac{1}{tr\left(\hat{T}\,\hat{T}\right)}\left[F^2 D + a^2\,\hat{T}\,tr\left(\hat{T}\,D\right)\right]$$
$$+ f_d\,a\,F\left(\hat{T} + \hat{T}^*\right)\|D\|,\qquad(4)$$

with $\hat{T} = T/trT$ and $\hat{T}^* = \hat{T} - \frac{1}{3}I$, where I is the 2nd order unit tensor and tr is the trace of a 2nd order tensor.

$$a = \sqrt{\frac{3}{8}\frac{\left(3 - \sin\phi_c\right)}{\sin\phi_c}}$$

$$F = \sqrt{\frac{1}{8}\tan^2\psi + \frac{2 - \tan^2\psi}{2 + \sqrt{2}\tan\psi\cos 3\Theta}} - \frac{1}{2\sqrt{2}}\tan\psi$$

$$\cos 3\Theta = -\sqrt{6}\frac{tr\left(\hat{T}^{*3}\right)}{\left[tr\left(\hat{T}^{*2}\right)\right]^{3/2}}\qquad \tan\psi = \sqrt{3}\left\|\hat{T}^*\right\|$$

The functions f_e, f_d and f_b take into account the influence of density and stress state on the material behavior.

$$f_e = \left(\frac{e_c}{e}\right)^{\beta}\qquad\qquad f_d = \left(\frac{e - e_d}{e_c - e_d}\right)^{\alpha}$$

$$f_b = \frac{\dfrac{h_s}{n}\left(\dfrac{1 + e_i}{e_i}\right)\left(\dfrac{e_{i0}}{e_{c0}}\right)^{\beta}\left(\dfrac{-trT}{h_s}\right)^{1-n}}{\left[3 + a^2 - \sqrt{3}\,a\left(\dfrac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}}\right)^{\alpha}\right]}$$

That hypoplastic model uses 8 material parameters - the critical friction angle ϕ_c, the granular hardness h_s, the limiting void ratios e_{c0}, e_{d0}, e_{i0} and the exponents n, α, β - all together to be determined mainly from standard soil mechanical laboratory tests (Herle and Gudehus (1999)).

The hypoplastic model for sand (von Wolffersdorff (1996)) was extended by Niemunis and Herle (1997) in order to simulate the mechanical behaviour under cyclic loading conditions. By including a so-called "intergranular strain", which serves as an additional state variable, the stiffness is modified dependant on the direction of the previous strain increment.

Numerical Implementation

In general the numerical implementation of hypoplastic constitutive equations follows the same procedures as for classical elastoplastic models. But because of the fact that there are no surfaces explicitly defined, there is no need to develop the numerical solution in respect to any such yield or potential surface. A standard numerical integration scheme, as described above, can be applied instead, to integrate the constitutive equations for each time step during the numerical solution process. That makes the numerical implementation of hypoplastic models much easier and leads to quite robust numerical behavior (Rackwitz (2003), Kolymbas (2000)).

Numerical Simulation of Element Tests

The above described hypoplastic soil model by von Wolffersdorff (1996) was also implemented into the general purpose FE-Program *ANSYS* and the parameters for that constitutive model were evaluated for Berlin sand (Rackwitz (2003)). The data of triaxial compression tests with Berlin sand can now be used to check the quality of the used hypoplastic constitutive equation.

The comparison of laboratory and numerical results from drained triaxial compression tests at a confining cell pressure p_0 = 500 kPa is shown in Fig 8. Starting from three different initial relative densities D_r, ranging from loose to dense, the stress-strain paths of all tests reach a joint critical state after monotonic shearing up to about 21% axial strain (Fig 8 top).The

hypoplastic model also represents typical volumetric behavior of sand (Fig 8 bottom). Dilatancy of initially medium dense to dense sand is well predicted. The results for initially loose samples (D_r = 13%) are less accurate.
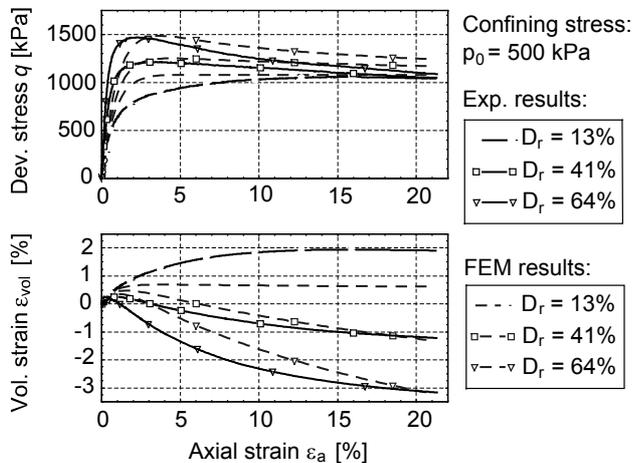

Fig 8: Experimental and numerical results from drained triaxial compression tests with Berlin sand

OUTLOOK

The lecture continues with the application of different approaches to the one-dimensional shear wave propagation. The computer program *SHAKE91* will be used for the calculations including an equivalent linear viscoelastic model and the FE-code *ANSYS* for the truly non-linear *CSSA*-Model. Since pure (bending-free) shearing can not be handled by the majority of one-dimensional finite elements, plane distorted quadrilateral elements in combination with special boundary conditions will be used.

The response of both approaches undergoing base excitation will be compared. In contrast to the equivalent linear model, the *CSSA*-Model, due to its concept of state dependent dilatancy, will show some density development and vertical displacements of the ground surface under drained conditions. Under undrained conditions, a pore pressure build up in the soil profile may occur, followed by a decrease of the effective stresses. Thus, conclusions can be drawn about the cyclic mobility of the sand deposit or the exposure to liquefaction. Finally, these results will point out the capabilities of advanced constituitive equations for soils in earthquake engineering.

REFERENCES

ANSYS Inc. (2000) *Guide to ANSYS 5.6 User Programmable Features*

Aubram, D. (2004) *Implementation of an Appropriate 'Stress Point Algorithm' for the CSSA-Model in ANSYS*, Diploma Thesis, Soil Mechanics and Foundation Engineering Division, Technical University of Berlin (TU Berlin) [in German]

Dafalias, Y.F. (1986) "Bounding Surface Plasticity I: Mathematical Foundation and Hypoplasticity", *Journal of Engineering Mechanics*, Vol. 112, No. 9, pp. 966-987

Gudehus, G. (1996) "A Comprehensive Constitutive Equation for Granular Materials", *Soils and Foundations*, Vol. 36, No. 1, pp. 151-190

Herle, I. and Gudehus, G. (1999) "Determination of Parameters of a Hypoplastic Constituitve Model from Properties of Grain Assembles", *Mechanics of Cohesive-Frictional Materials*, Vol. 4, pp. 461-486

Ishihara, K., Tatsuoka, F. and Yasuda, S. (1975) "Undrained Deformation and Liquefaction of Sand under Cyclic Stresses", *Soils and Foundations*, Vol. 15, No. 1, pp. 29-44

Kolymbas, D. (2000) "Hypoplasticity - A New Paradigma for Plasticity", *Anniversary Volume Krzysztof Wilmanski – Contributions to Continuum Theories, WIAS-Report No. 18*, Berlin, pp. 129-137

Kolymbas, D., Herle, I., Von Wolffersdorff, P.-A. (1995) "Hypoplastic Constitutive Equation with Internal Variables", *International Journal for Numerical and Analytical Methods in Geomechanics*, Vol. 19, pp. 415-436

Kolymbas, D., Wu, W. (1993) "Introduction to Hypoplasticity", *Proceedings of a Workshop held in Horton, Greece, 12-16 June 1992, Modern Approaches to Plasticity*, D. Kolymbas (ed.), Elsevier, pp. 213-224

Li, X.S. (1998) "Linear Representation of Steady-State Line", *Gjournal of Geotechnical and Geoenvironmental Engineering*, Vol. 124, No. 12, pp. 1215-1217

Li, X.S. (1999) "Implementation of CSSA-Model into the FE-Code SUMDES", Personal Communication

Li, X.S. (2002) "A Sand Model with State Dependent Dilatancy", *Géotechnique*, Vol. 52, No. 3, pp. 173-186

Li, X.S. and Dafalias, Y.F. (2000) "Dilatancy for Cohesionless Soils", *Géotechnique*, Vol. 50, No. 4, pp. 449-460

Niemunis, A. and Herle, I. (1997) "Hypoplastic Model for Cohesionless Soils with Elastic Strain Range", *Mechanics of Cohesive-Frictional Materials*, Vol. 2, pp. 279-299

Pastor, M., Zienkiewicz, O.C. and Chan, A.H.C. (1990) "Generalized Plasticity and the Modelling of Soil Behaviour", *International Journal for Numerical and Analytical Methods in Geomechanics*, Vol. 14, pp. 151-190

Rackwitz, F. (2003) *Numerical Analysis of Single Tension Piles and Groups of Tension Piles in Sand Based on Loading Tests*, Dissertation, Soil Mechanics and Foundation Engineering Division Series, No. 32, Technical University of Berlin (TU Berlin), S.A. Savidis (ed.) [in German]

Roscoe, K.H., Schofield, A.N. and Wroth, C.P. (1958) "A Sand Model with State Dependent Dilatancy", *Géotechnique*, Vol. 8, No. 1, pp. 22-53

Von Wolffersdorff, P.-A. (1996) "A Hypoplastic Relation for Granular Materials with a Predefined Limit State", *Mechanics of Cohesive-Frictional Materials*, Vol. 1, pp. 251-271